

The Least Squares Klein Bottle for Image Patches

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Outline

The Klein Bottle in Image Patches

A General Formulation for Image Patches
Examples

Finding an optimal model

Extensions of the Klein Bottle Model

Testing the Klein Bottle Hypothesis

Conclusion

History

- ▶ Lee et al. "The Nonlinear Statistics of Image Patches"
- ▶ Carlsson & de Silva use as an example in witness Complexes
- ▶ Carlsson paper on image patches
- ▶ BiWedgelets
- ▶ Perea - pattern recognition
- ▶ Adams - range images primary circle

Some Questions

- ▶ How are patches distributed on the Klein bottle model? (Carlsson, Perea)
- ▶ What are good representations of the Klein bottle? (This work)
- ▶ What higher order structures can be built on top of the Klein Bottle?
- ▶ Does the Klein bottle really account for the "interesting" topological features of the data?

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Whence the Klein Bottle?



Klein Bottle artist: Ron Estrin

Edges

Our model for “high-contrast” image patches is based on the observation that patches that look like edges are common.
(here) figure of edge in picture

Elements

Two sources of variation in edges:

- ▶ Orientation of edges (primary circle)
- ▶ Type of edge (secondary circles)

(here) pictures

Odd Functions

Odd functions capture the behavior of transitions from one intensity to another

Even Functions

Even functions capture the behavior of lines

Combined Functions

Edges can also be a combination of even and odd functions

Klein Bottle

Two elements:

- ▶ Family of even and odd functions on a symmetric domain parameterized by the circle, and with the property $f(\theta) = -f(\theta)$.
- ▶ Action of S^1 on each function (rotation)

Klein Bottle

Identification:

Klein Bottle

(proof)

Discrete Grids

2D DCT (diagonalizes normalization operator)

Polynomials on Patches

Another idea: even and odd functions evaluated at points on the grid

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Least Squares Formulation

Now, we have many ways to explain how the Klein bottle may appear in image patches. Which one best fits the data?

Consider a family of Klein bottle models parameterized by α , with Klein bottle K_α , and projection operation

$$P_\alpha x = \arg \inf_{f \in K_\alpha} \|x - f\|$$

We would like to find the parameter α that minimizes

$$\min_{\alpha} \sum_{x \in X} \|x - P_\alpha x\|^2$$

Empirical Results

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Covering Tori

What happens if edges are not perfectly centered?

End up with line bundle over the Klein bottle Torus is a section of the bundle (can't make the Klein bottle identification)

Non-normalized data

Cone with constant patch

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Algebraic Mapping Cone

Model Selection & Model Testing

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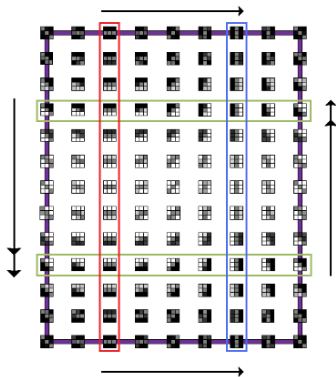
Conclusion

Ongoing/Future Work

Questions

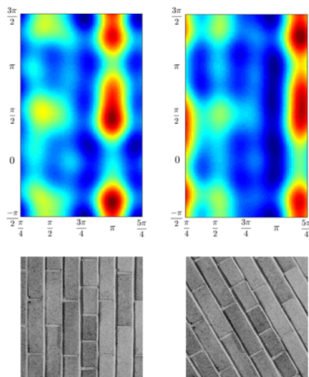
References

Patch Space



J. Perea, G. Carlsson “A Klein-Bottle-Based Dictionary for Texture Representation” (2014)

Texture Recognition



J. Perea, G. Carlsson “A Klein-Bottle-Based Dictionary for Texture Representation” (2014)

Compression



Wedgelets



Adaptive wedgelets



Adaptive bi-wedgelet



A. Maleki, M. Shahram, G. Carlsson “A Near Optimal Coder For Image Geometry With Adaptive Partitioning” (2008)

Interesting Questions

- ▶ What are higher order structures built on top of the Klein Bottle?
- ▶ What are optimal representations of the Klein bottle (in the image patch setting)?
- ▶ What are optimal bases to use in representing patch densities?